SUGGESTED SOLUTIONS TO HOMEWORK 6

Exercise 1 (7.2.4)**.** If $\alpha(x) := -x$ and $\omega(x) := x$ and if $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in [0,1]$, does it follow from the squeeze Theorem 7.2.3 that $f \in \mathcal{R}[0,1]$?

Proof. No. Consider

$$f(x) := \begin{cases} x, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$$

Then $f \notin \mathcal{R}[0,1]$. Indeed, let $\dot{P}_n := \{([\frac{i-1}{n}, \frac{i}{n}], \frac{2i-1}{2n})\}_{i=1}^n$ and $\dot{Q}_n := \{([\frac{i-1}{n}, \frac{i}{n}], \frac{2(i-1)+\sqrt{2}}{2n})\}_{i=1}^n$, then $\|\dot{P}_n\| = \|\dot{Q}_n\| = \frac{1}{n}$ and we have

$$S(f;\dot{P}_n) = \sum_{i=1}^n \frac{2i-1}{2n^2} = \frac{1}{2}, \quad S(f;\dot{Q}_n) = 0,$$

which implies that for arbitrary $\eta > 0$, there exist two partitions $\dot{P}_n$ and $\dot{Q}_n$ with $n := [\eta^{-1}]+1$ such that $|S(f,\dot{P}_n) - S(f,\dot{Q}_n)| = 1/2$. Therefore, Theorem 7.2.3 is not applicable. □

Exercise 2 (7.2.8)**.** Suppose that f is continuous on $[a,b]$, that $f(x) \ge 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Proof. Let us prove by contradiction. Assume there exist $c \in (a,b)$ such that $f(c) > 0$, then by continuity of f, there exists $\delta > 0$ such that

$$f(x) > \frac{1}{2} f(c),$$

for $|x - c| < \delta$. Then

$$\int_a^b f \ge \int_{c-\delta}^{c+\delta} f \ge f(c)\delta > 0,$$

which is a contradiction. □

Exercise 3 (7.2.9)**.** Show that the continuity hypothesis in preceding exercise cannot be dropped.

Proof. Consider

$$f(x) := \begin{cases} 1, & x = a, \\ 0, & x \ne a. \end{cases}$$

Then $\int_a^b f = 0$. Indeed, for arbitrary $\varepsilon > 0$, let $0 < \delta < \varepsilon$, then for all partitions $\dot{P}$ with $\|\dot{P}\| < \delta$, we have

$$|S(f;\dot{P})| < \delta < \varepsilon.$$

However, $f(x) \ne 0$ for all $x \in [a,b]$. □

Exercise 4 (7.2.11)**.** If f is bounded by M on $[a,b]$ and if the restriction of f to every interval $[c,b]$ where $c \in (a,b)$ is Riemann integrable, show that $f \in \mathcal{R}[a,b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$.

Proof. Let $\varepsilon > 0$, let us define

$$\alpha_\varepsilon(x) := \begin{cases} -M, & a \le x \le a + \frac{\varepsilon}{4M}, \\ f(x), & a + \frac{\varepsilon}{4M} < x \le b. \end{cases} \quad \omega_\varepsilon := \begin{cases} M, & a \le x \le a + \frac{\varepsilon}{4M}, \\ f(x), & a + \frac{\varepsilon}{4M} < x \le b. \end{cases}$$

Then we have $\alpha_\varepsilon \le f \le \omega_\varepsilon$. Moreover,

$$\left| \int_a^b \alpha_c - \int_a^b \omega_c \right| \le \frac{\varepsilon}{2} < \varepsilon.$$

Therefore by Squeeze theorem, we have $f \in \mathcal{R}[a, b]$.

Let $a < c < a + \frac{\varepsilon}{2M}$, then

$$\left| \int_c^b f - \int_a^b f \right| = \left| \int_a^c f \right| \le M(c-a) \le \frac{\varepsilon}{2} < \varepsilon,$$

which implies that $\int_c^b f \to \int_a^b f$ as $c \to a+$. □

Exercise 5 (7.2.12)**.** Show that $g(x) := \sin(1/x)$ for $x \in (0, 1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0, 1]$.

Proof. Since $|g| \le 1$ and g is continuous on $[c, 1]$ for $c > 0$, which implies $g \in \mathcal{R}[c, 1]$, by the above Exercise 4, we have $f \in \mathcal{R}[0, 1]$. □

Exercise 6 (7.2.14)**.** Suppose that $f : [a, b] \to \mathbb{R}$, that $a = c_0 < c_1 < \cdots < c_m = b$ and that the restrictions of f to $[c_{i-1}, c_i]$ belong to $\mathcal{R}[c_{i-1}, c_i]$ for $i = 1, ..., m$. Prove that $f \in \mathcal{R}[a, b]$ and that the formula in Corollary 7.2.11 holds.

Proof. We use mathematical induction on $m \in \mathbb{N}$.

For $m = 1$, the conclusion is trivial. For $m = 2$, the conclusion follows by Additivity theorem.

Let us assume the conclusion is true for $m \ge k$ for $k \in \mathbb{N}$. For $m = k + 1$, by assumption, $f \in \mathcal{R}[a, c_k]$, moreover, using Additivity theorem again, since $f \in \mathcal{R}[c_k, b]$, we have $f \in \mathcal{R}[a, b]$. In addition, we have

$$\begin{aligned} \int_a^b f &= \int_a^{c_k} f + \int_{c_k}^b f \\ &= \sum_{i=1}^{k} \int_{c_{i-1}}^{c_i} f + \int_{c_k}^b f \\ &= \sum_{i=1}^{k+1} \int_{c_{i-1}}^{c_i} f. \end{aligned}$$

which implies the conclusion holds for $m = k + 1$. Therefore by mathematical induction, the conclusion is true for all $m \in \mathbb{N}$. □